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Persistence in advection of a passive scalar

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We consider the persistence phenomenon in advected passive scalar equation in one dimension. The velocity field is random with the $\langle v(k,\omega)v(-k,-\omega)\rangle \sim |k|^{-(2+\alpha)}$. In the presence of the nonlinearity the complete Green's function becomes $G^{-1}=-i\omega+Dk^2+\Sigma$. We determine Σ self-consistently from the correlation function which gives $\Sigma \sim k^{\beta}$, with $\beta=(1-\alpha)/2$. The effect of the nonlinear term in the equation in the $\mathcal{O}(\epsilon^2)$ is to replace the diffusion term due to molecular viscosity by an effective term of the form $\Sigma_0 k^{\beta}$. The stationary correlator for this system is $[\text{sech}(T/2)]^{1/\beta}$. Using the self-consistent theory we have determined the relation between β and α. Finally, the independent interval approximation (IIA) method is used to determine the persistent exponent.

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The problem of persistence or survival [1] has attracted a lot of interest in the last decade. The persistence probability has been obtained both analytically and numerically for a large class of stochastic process, Markovian as well as non-Markovian. The random walk problem, diffusion problem, surface growth, and Ising model with Glauber dynamics are only a few [1–24]. The persistence probability or the zero crossing probability is simply the probability that the local field $\phi(x,t)$ has not changed sign up to time t. For single particle systems such as the random walker, which are also Markovian in nature, the persistence probability is easy to calculate since the stationary correlator of such a process decays exponentially for all times. For many body systems where the field ϕ has a space dependence the calculation of the zero crossing probability becomes complicated. The problem is now twofold—first we have to write down an effective equation for a single site process by solving the underlying dynamics of the many particle system and then from this effective single site equation we have to find out the persistence probability. Even though the first part is achieved, the second part of obtaining the persistence probability is notoriously tough since the resulting single site process becomes non-Markovian.

The simplest of such a process which one can think of is the diffusion equation $\partial_t\phi=D\partial_x^2\phi$. The fact that this is an effective single site equation can be seen from the solution $\phi(x,t)=\int dx' G(x-x',t)\phi(x',0)$, where $G(x,t)$ is the Green's function for diffusion equation. The problem of persistence in a diffusion equation has already been addressed by Majumdar *et al.* [2]. They considered the diffusion equation with random initial condition $\phi(x,0)$ taken from a Gaussian distribution. The two time correlation function $C(t_1,t_2)$ of the normalized variable $X=\phi(x,t)/\sqrt{\langle\phi^2(x,t)\rangle}$ takes the form

$$C(t_1,t_2) \equiv \langle X(t_1)X(t_2)\rangle = [4t_1t_2/(t_1+t_2)^2]^{D/4}, \tag{1}$$

where D is the dimension of space. Now if we make the transformation $T=\ln t$, the correlator $C(T_1,T_2)$ becomes $f(|T_1-T_2|)$, with $f(T)=[\text{sech}(T/2)]^{D/2}$, which is clearly stationary. The stationary correlator for the effective single site process is not exponentially decaying and therefore the calculation of the persistence exponent becomes difficult. The fact that the correlator is not exponentially decaying indicates that the effective single site process is non-Markovian because of the interaction with nearest neighbor sites. Given this stationary non-Markovian correlator it then remains to determine the persistence probability. Two methods have been developed to address this problem, the independent interval approximation (IIA) [2] and the "series expansion" [9] approach. In this present article we will use IIA to evaluate the exponents.

Knowing the information about the persistence exponents for a diffusive process, it is natural to ask what would be zero crossing probability when the diffusive process is augmented by an advection term. For a simple diffusive process, if L is the relevant length scale (say the size of the container), then the time to diffuse to a distance L is simply $\tau_1=L^2/2D$. If, however, the particles are advected then the time for them to diffuse through a distance L is $\tau_2=L/v$, where v is the advection velocity. The ratio of the two time scales is

$$\tau_1/\tau_2 = 2S/\text{Re}, \tag{2}$$

where S is the Schmidt number and Re is the Reynolds number. The Schmidt number is of the order of unity and therefore, it follows that the mixing time due to advection is smaller than the pure diffusive process. We therefore expect that the exponents will be greater than those for the pure diffusive process.

The advected passive scalar equation reads

$$\frac{\partial\phi}{\partial t} + \vec{v}(\vec{x},t)\cdot\vec{\nabla}\phi = D\nabla^2\phi \tag{3}$$

together with

$$\vec{\nabla}\cdot\vec{v}=0. \tag{4}$$

In one dimension however, the constraint imposed by Eq. (4) is relaxed, since imposing Eq. (4) in one dimension would mean that the velocity is constant. Instead, in one dimension, we will consider a random velocity field drawn from a Gaussian distribution. In particular, the velocity-velocity correlation is given by

$$\langle v(x,t)v(x',t')\rangle = 2D_1 g(|x-x'|)\delta(t-t'). \tag{5}$$

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The Fourier transform of Eq. (3) in one dimension is

$$\frac{\partial\tilde{\phi}}{\partial t}+i\int\frac{dp}{2\pi}p\tilde{v}(k-p,t)\tilde{\phi}(p,t)=Dk^2\tilde{\phi}(k,t) \quad (6)$$

and the velocity-velocity correlation transforms to

$$\langle\tilde{v}(k,t)\tilde{v}(k',t)\rangle=2D\tilde{g}(|k|)\delta(k+k')\delta(t-t'), \quad (7)$$

where $\tilde{\phi}(k,t)$, $\tilde{v}(k,t)$, and $\tilde{g}(k)$ are the Fourier transform of $\phi(x,t)$, $v(x,t)$, and $g(x)$, respectively. We choose $\tilde{g}(|k|)$ as a power law decaying function, that is,

$$\tilde{g}(|k|)=\frac{1}{|k|^{(2+\alpha)}}, \quad (8)$$

with $0<\alpha<1$. Since for $k=0$, $g(|k|)$ diverges, we shall set $\tilde{v}(0)=0$.

In absence of the nonlinear term the Green's function for Eq. (6) is

$$G_0^{-1}=-i\omega+Dk^2. \quad (9)$$

The effect of the nonlinearity is to replace the zeroth order Green's function by its most general form,

$$G^{-1}=-i\omega+Dk^2+\Sigma=G_0^{-1}+\Sigma. \quad (10)$$

Hence,

$$G=\frac{G_0}{1+G_0\Sigma}=G_0(1-\Sigma G_0+\Sigma^2G_0^2+\cdots). \quad (11)$$

The correlation function $\langle\tilde{\phi}(k,\omega)\tilde{\phi}(-k,-\omega)\rangle$ can be written as

$$\langle\tilde{\phi}(k,\omega)\tilde{\phi}(-k,-\omega)\rangle\sim GG$$
$$=G_0G_0-G_0\Sigma G_0+G_0\Sigma^2G_0^2+\cdots. \quad (12)$$

We will determine the self-energy Σ self-consistently from the correlation function.

Taking a Fourier transformation of Eq. (6) in time domain we obtain

$$-i\omega\tilde{\phi}(k,\omega)+i\int\frac{dp}{2\pi}\frac{d\omega'}{2\pi}p\tilde{v}(k-p,\omega-\omega')\tilde{\phi}(p,\omega')$$
$$=-Dk^2\tilde{\phi}(k,\omega), \quad (13)$$

while the velocity correlation function becomes

$$\langle\tilde{v}(k,\omega)\tilde{v}(k',\omega')\rangle=2D\tilde{g}(|k|)\delta(k+k')\delta(\omega+\omega'). \quad (14)$$

We now make a perturbative expansion in ϕ and write

$$\tilde{\phi}=\tilde{\phi}_0+\epsilon\tilde{\phi}_1+\epsilon^2\tilde{\phi}_2+\cdots. \quad (15)$$

Substituting this in Eq. (13), the zeroth order solution is

$$\tilde{\phi}_0(k,\omega)[-i\omega+Dk^2]=\tilde{\phi}_0(k,0),$$

$$\tilde{\phi}_0(k,\omega)=\frac{\tilde{\phi}_0(k,0)}{[-i\omega+Dk^2]}=G_0(k,\omega)\tilde{\phi}_0(k,0). \quad (16)$$

As in the case of normal diffusion we choose random initial condition $\phi_0(k,0)$ drawn from a Gaussian distribution with the correlation

$$\langle\tilde{\phi}_0(k,0)\rangle=0,$$

$$\langle\tilde{\phi}_0(k,0)\tilde{\phi}_0(k',0)\rangle=2D_2\delta(k+k'). \quad (17)$$

In the first order the solution for $\phi_1(k,\omega)$ is

$$\tilde{\phi}_1(k,\omega)=G_0(k,\omega)\left(-i\int\frac{dp}{2\pi}\frac{d\omega'}{2\pi}p\tilde{v}\right.$$
$$\left.\times(k-p,\omega-\omega')\tilde{\phi}_0(p,\omega')\right) \quad (18)$$

while the solution for $\tilde{\phi}_2$ becomes

$$\tilde{\phi}_2(k,\omega)=G_0(k,\omega)\left(-i\int\frac{dp}{2\pi}\frac{d\omega'}{2\pi}p\tilde{v}\right.$$
$$\left.\times(k-p,\omega-\omega')\tilde{\phi}_1(p,\omega')\right)$$
$$=G_0(k,\omega)\left[-i\int\frac{dp}{2\pi}\frac{d\omega'}{2\pi}p\tilde{v}(k-p,\omega-\omega')G_0(p,\omega')\right.$$
$$\left.\times\left\{-i\int\frac{dq}{2\pi}\frac{d\omega''}{2\pi}q\tilde{v}(p-q,\omega'-\omega'')\tilde{\phi}_0(q,\omega'')\right\}\right]. \quad (19)$$

To evaluate Σ self-consistently we need to calculate the correlation function $\langle\tilde{\phi}_1(k,\omega)\tilde{\phi}_1(-k,-\omega)\rangle\langle\tilde{\phi}_2(k,\omega)\tilde{\phi}_0(-k,-\omega)\rangle$. We assume that the nonlinear contribution to the total Green's function G will dominate over the Dk^2 term [25]. Hence, we rewrite Eq. (10) as

$$G^{-1}=-i\omega+\Sigma, \quad (20)$$

which shows that ω and Σ have the same dimension.

The correlation $\langle\tilde{\phi}_2(k,\omega)\tilde{\phi}_0(-k,-\omega)\rangle$ is then

$$\langle\tilde{\phi}_2(k,\omega)\tilde{\phi}_0(-k,-\omega)\rangle$$
$$=-G_0(k,\omega)\left\langle\left[\int\frac{dp}{2\pi}\frac{d\omega'}{2\pi}p\tilde{v}(k-p,\omega-\omega')G_0(p,\omega')\right.\right.$$
$$\times\left\{\int\frac{dq}{2\pi}\frac{d\omega''}{2\pi}q\tilde{v}(p-q,\omega'-\omega'')\tilde{\phi}_0(q,\omega'')\right\}$$
$$\left.\left.\times\tilde{\phi}_0(-k,-\omega)\right]\right\rangle. \quad (21)$$

A little algebra simplifies the above expression to

$$\langle\tilde{\phi}_2(k,\omega)\tilde{\phi}_0(-k,-\omega)\rangle = -D_2G_0(k,\omega)\left[\int\frac{dp}{2\pi}\frac{d\omega'}{2\pi}\frac{d\omega''}{2\pi}\times kp\langle\tilde{v}(k-p,\omega-\omega')\tilde{v}(p-k,\omega'-\omega'')\rangle\times G_0(k,\omega'')G_0(-k,-\omega)\right]. \quad (22)$$

The velocity-velocity correlation gives a $\delta(\omega-\omega'')$ which, after the ω'' integral becomes

$$\langle\tilde{\phi}_2(k,\omega)\tilde{\phi}_0(-k,-\omega)\rangle = -D_1D_2G_0(k,\omega)\left[\int\frac{dp}{2\pi}\frac{d\omega'}{2\pi}\frac{kp}{|k-p|^{(2+\alpha)}}G_0(p,\omega')\right]\times G_0(k,\omega)G_0(-k,-\omega). \quad (23)$$

2We now turn our attention to $\langle\tilde{\phi}_1(k,\omega)\tilde{\phi}_1(-k,-\omega)\rangle$ which is given by

$$\langle\tilde{\phi}_1(k,\omega)\tilde{\phi}_1(-k,-\omega)\rangle = -G_0(k,\omega)\left[\int\frac{dp}{2\pi}\frac{d\omega'}{2\pi}\frac{dq}{2\pi}\frac{d\omega''}{2\pi}pq\langle\tilde{v}(k-p,\omega-\omega')\times\tilde{v}(-k-q,-\omega-\omega')\rangle\langle\tilde{\phi}_0(p,\omega')\tilde{\phi}_0(q,\omega'')\rangle\right]\times G_0(-k,-\omega). \quad (24)$$

The velocity-velocity correlation introduces a $\delta(p+q)\delta(\omega'+\omega'')$ while the average $\langle\tilde{\phi}_0(p,\omega')\tilde{\phi}_0(q,\omega'')\rangle$ gives us $\delta(p+q)$. Integrating over the q and ω'' variables we obtain

$$\langle\tilde{\phi}_1(k,\omega)\tilde{\phi}_1(-k,-\omega)\rangle = D_1D_2G_0(k,\omega)\left[\int\frac{dp}{2\pi}\frac{d\omega'}{2\pi}\frac{p^2}{|k-p|^{(2+\alpha)}}G_0(p,\omega')\times G_0(-p,-\omega')\right]G_0(-k,-\omega). \quad (25)$$

The second term in Eq. (12) has the same structure of $\langle\tilde{\phi}_1(k,\omega)\tilde{\phi}_1(-k,-\omega)\rangle$ while the third term has the same structure as $\langle\tilde{\phi}_2(k,\omega)\tilde{\phi}_0(-k,-\omega)\rangle$. Thus Eq. (23) gives us

$$\Sigma^2 \sim \int\frac{dp}{2\pi}\frac{d\omega'}{2\pi}\frac{kp}{|k-p|^{(2+\alpha)}}G_0(p,\omega') \quad (26)$$

or

$$\Sigma \sim k^{(1-\alpha)/2}, \quad (27)$$

while from Eq. (25) we obtain

$$\Sigma \sim \int\frac{dp}{2\pi}\frac{d\omega'}{2\pi}\frac{p^2}{|k-p|^{(2+\alpha)}}\frac{1}{\omega'^2+D^2p^4}. \quad (28)$$

Since $\omega\sim\Sigma$, and neglecting the Dp^2 term compared to Σ, power counting yields

$$\Sigma \sim \frac{k^{(1-\alpha)}}{\Sigma}, \quad (29)$$

which gives us the same result as in Eq. (27). We remark, in passing, that the result obtained in Eq. (27) can also be obtained by introducing a noise term in Eq. (6). It should be noted that for a Kolmogorov-like velocity field, $\alpha=-1/3$.

Before we proceed to calculate the exponent with the effective equation, we need to clarify the Gaussian nature of ϕ, since it is crucial for IIA to work. For this we calculate the correlations $\phi_1(k,\omega)\phi_1(-k,-\omega)\phi_1(k,\omega)\phi_1(-k,-\omega)$ and $\phi_2(k,\omega)\phi_0(-k,-\omega)\phi_2(k,\omega)\phi_0(-k,-\omega)$,

$$\langle\phi_1(k,\omega)\phi_1(-k,-\omega)\phi_1(k,\omega)\phi_1(-k,-\omega)\rangle = G_0(k,\omega)G_0(-k,-\omega)G_0(k,\omega)G_0(-k,-\omega)\times\left[\int\frac{dp_1}{2\pi}\frac{d\omega_1}{2\pi}\frac{dp_2}{2\pi}\frac{d\omega_2}{2\pi}\frac{dp_3}{2\pi}\frac{d\omega_3}{2\pi}\frac{dp_4}{2\pi}\frac{d\omega_4}{2\pi}p_1p_2p_3\times\langle\tilde{v}(k-p_1,\omega-\omega_1)\tilde{v}(-k-p_2,-\omega-\omega_2)\times\tilde{v}(k-p_3,\omega-\omega_3)\tilde{v}(-k-p_4,-\omega-\omega_4)\rangle\times\langle\tilde{\phi}(p_1,\omega_1)\tilde{\phi}(p_2,\omega_2)\tilde{\phi}(p_3,\omega_4)\tilde{\phi}(p_4,\omega_4)\rangle\right].$$

Since both $\tilde{v}$ and $\tilde{\phi}$ are Gaussian we can decompose their four point correlations as product of two point correlations. There are two possibilities. In the first case $k-p_1$ couples with $-k-p_4$ and $k-p_3$ couples with $-k-p_2$, while the second one is when $k-p_1$ couples with $k-p_3$ and $-k-p_2$ couples with $-k-p_4$. It can be shown, using the fact that $\tilde{v}(0)=0$, the contribution to the four point correlation function from the second case is zero. In the first case when $k-p_1$ couples with $-k-p_4$ and $k-p_3$ couples with $-k-p_2$ we note that $\tilde{\phi}(p_1,\omega_1)$ couples with $\tilde{\phi}(p_4,\omega_4)$ while $\tilde{\phi}(p_3,\omega_3)$ couples with $\tilde{\phi}(p_2,\omega_2)$. In which case the four point correlation function becomes

$$\langle\phi_1(k,\omega)\phi_1(-k,-\omega)\phi_1(k,\omega)\phi_1(-k,-\omega)\rangle \sim G_0(k,\omega)\times G_0(-k,-\omega)G_0(k,\omega)G_0(-k,-\omega)\left[\int\frac{dp_1}{2\pi}\frac{d\omega_1}{2\pi}\frac{dp_3}{2\pi}\frac{d\omega_3}{2\pi}\times\frac{p_1^2}{|k-p_1|^{2+\alpha}}\frac{p_3^2}{|k-p_3|^{2+\alpha}}G_0(p_1,\omega_1)G_0(-p_1,-\omega_1)\times G_0(p_3,\omega_3)G_0(-p_3,-\omega_3)\right],$$

which is clearly the square of the two point correlation function.

The correlation $\langle\phi_2(k,\omega)\phi_0(-k,-\omega)\phi_2(k,\omega)\phi_0(-k,-\omega)\rangle$ is

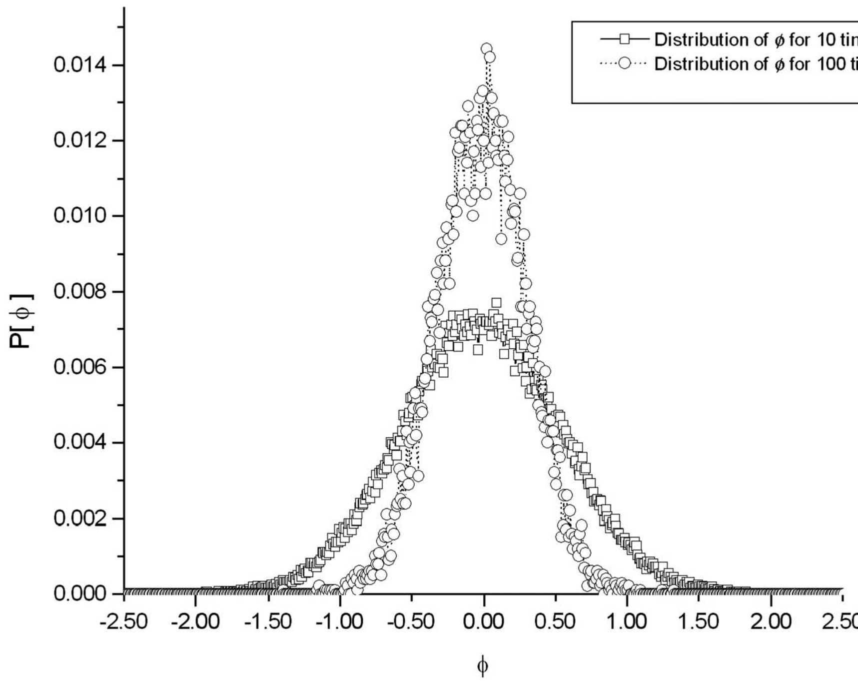

FIG. 1. Plot of probability distribution of $\phi(0,t)$, $P[\phi]$. The square points are for 10 time steps while the circle points are for 100 time steps. A configuration of 10^5 has been done to obtain the data for 10 time steps while a configuration of 10^4 has been done to obtain the data for 100 time steps.

$$\langle \phi_2(k,\omega)\phi_0(-k,-\omega)\phi_2(k,\omega)\phi_0(-k,-\omega)\rangle$$
$$= G_0(k,\omega)G_0(k,\omega)\int \frac{dp_1}{2\pi}\frac{d\omega_1}{2\pi}\frac{dp_2}{2\pi}\frac{d\omega_2}{2\pi}\frac{dp_3}{2\pi}\frac{d\omega_3}{2\pi}\frac{dp_4}{2\pi}\frac{d\omega_4}{2\pi}$$
$$\times p_1\tilde{v}(k-p_1,\omega-\omega_1)G_0(p_1,\omega_1)p_2\tilde{v}$$
$$\times(p_1-p_2,\omega_1-\omega_2)\phi_0(p_2,\omega_2)$$
$$\times p_3\tilde{v}(k-p_3,\omega-\omega_3)G_0(p_3,\omega_3)p_4\tilde{v}$$
$$\times(p_3-p_4,\omega_3-\omega_4)\phi_0(p_3,\omega_4)\phi_0(-k,-\omega)\phi_0(-k,-\omega).$$

Now $\phi_0(p_2,\omega_2)$ couples with $\phi_0(-k,-\omega)$ while $\phi_0(p_3,\omega_3)$ couples with the other $\phi_0(-k,-\omega)$ giving $G_0(k,\omega)G_0(-k,-\omega)$ in both cases and $p_2=k$, $p_3=k$, $\omega_2=\omega$, and $\omega_3=\omega$. Hence, the four point correlation becomes

$$\langle \phi_2(k,\omega)\phi_0(-k,-\omega)\phi_2(k,\omega)\phi_0(-k,-\omega)\rangle$$
$$\sim G_0(k,\omega)\left[\int \frac{dp_1}{2\pi}\frac{d\omega_1}{2\pi}\frac{kp_1}{|k-p_1|^{2+\alpha}}G_0(p_1,\omega_1)\right]$$
$$\times G_0(k,\omega)G_0(-k,-\omega)$$
$$\times\left[\int \frac{dp_3}{2\pi}\frac{d\omega_3}{2\pi}\frac{kp_3}{|k-p_3|^{2+\alpha}}\right]G_0(p_3,\omega_3)$$
$$\times G_0(k,\omega)G_0(-k,-\omega),$$

which is once again the square of the two point correlation function. Since the fact that the four point correlation function for $\tilde{\phi}$ is simply the square of the two point correlation function, we conclude that the field $\tilde{\phi}$ remains Gaussian. This can be traced back to the fact that the probability distribution for $\tilde{\phi}$ and $\tilde{v}$ are Gaussian. We have also numerically found out the probability distribution $P[\phi]$ of $\phi(0,t)$ by simulating Eq. (3). The result is shown in Figs. 1 and 2, which indicates that $P[\phi]$ is Gaussian as anticipated.

Now, the persistence probability is simply the probability that the local field $\phi(x,t)$ has not changed sign for any x. Alternatively, since it is for any x, the question can also be framed in terms of the zero crossing probability of $\phi(0,t)$ defined as $\phi(0,t)=\int\frac{dk}{(2\pi)}\tilde{\phi}(k,t)$. It therefore suffices for IIA to work if the distribution of $\phi(0,t)$ is Gaussian. In Figs. 1 and 2, Eq. (3) has been simulated in the Fourier space and then $\phi(0,t)$ is obtained numerically integrating over the k space.

Thus, the effect of the nonlinearity in $\mathcal{O}(\epsilon^2)$ is to replace the term Dk^2 by an effective diffusion term that is $\Sigma_0 k^\beta$. We can, therefore, rewrite Eq. (6) as

$$\frac{\partial\tilde{\phi}}{\partial t} = -\Sigma_0 k^\beta\tilde{\phi} \quad (30)$$

with $\beta=(1-\alpha)/2$.

The two time correlation function $\langle\tilde{\phi}(k,\omega)\tilde{\phi}(-k,-\omega)\rangle$ becomes

$$\langle\tilde{\phi}(k,t_1)\tilde{\phi}(-k,t_2)\rangle = e^{-\Sigma_0 k^\beta(t_1+t_2)}. \quad (31)$$

The correlation $C(t_1,t_2)\equiv\langle\phi(x,t_1)\phi(x,t_2)\rangle$ for a fixed x (say $x=0$) is given by

$$C(t_1,t_2) = \int dk\langle\tilde{\phi}(k,t_1)\tilde{\phi}(-k,t_2)\rangle = \frac{1}{\beta}[\Sigma_0(t_1+t_2)]^{-1/\beta}. \quad (32)$$

It is important to ask how far in wave number space does the form of Eq. (30) extend given the fact that Eq. (27) is a low momentum result. We note that from Eq. (23) and Eq. (25) the amplitude of Σ is $\sqrt{D_1D_2/D}$, which determines the range of wave number space in which the contribution of Eq. (27) will dominate the background rate given by Dk^2. For $\sqrt{D_1D_2/D}$ of $\mathcal{O}(1)$ there will be significant range of k values

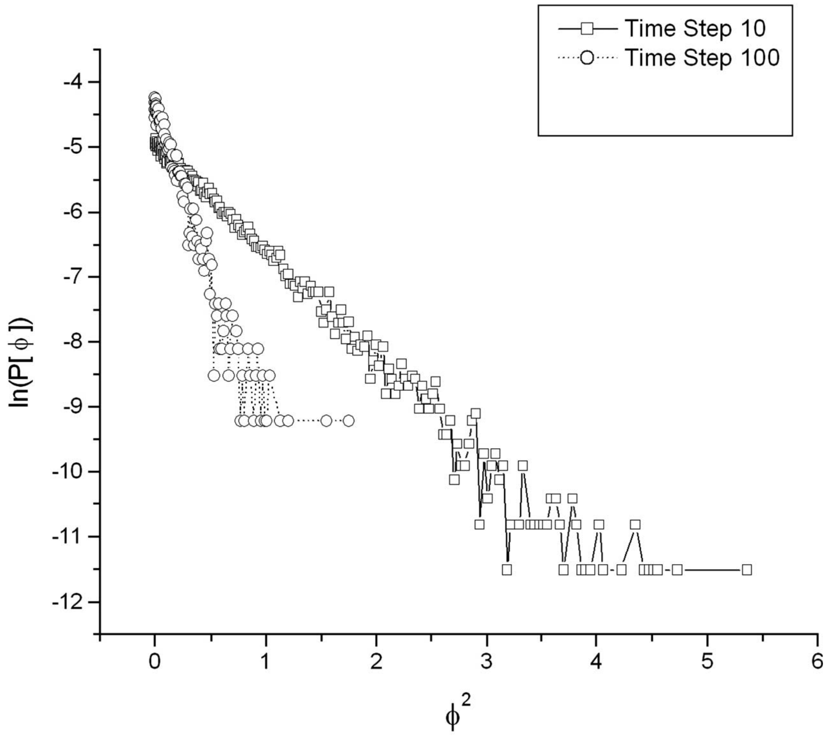

FIG. 2. Plot of the probability distribution $\ln(P[\phi])$ vs ϕ^2. The square points are 10 time steps while the circle points are 100 time steps. A configuration of 10^5 has been done to obtain the data for 10 time steps while a configuration of 10^4 has been done to obtain the data for 100 time steps.

for which k^{β} will dominate. Looking at Eq. (31), the dominant contribution to the integral comes from low values of k and hence the result found in Eq. (32) will dominate and give the asymptotically correct behavior.

An alternative explanation to the problem can be given by considering the complete correlator containing the Dk^2 term which is

$$\langle \tilde{\phi}(k,t_1)\tilde{\phi}(-k,t_2)\rangle = e^{-\Sigma_0 k^{\beta} t_2(1+t_1/t_2)-Dk^2 t_2(1+t_1/t_2)}.$$

Keeping the ratio t_1/t_2 fixed and considering t_1 and t_2 large it is clear that small k values will contribute, or else the correlation would decay to zero. Hence, asymptotically Eq. (32) would give the correct result.

Define the normalized variable $X(t)=\phi(x,t)/\sqrt{\langle \phi^2(x,t)\rangle}$. Then, the correlation $\langle X(t_1)X(t_2)\rangle$ in terms of $C(t_1,t_2)$ becomes

$$\bar{C}(t_1,t_2) \equiv \langle X(t_1)X(t_2)\rangle = C(t_1,t_2)/\sqrt{C(t_1,t_1)C(t_2,t_2)} = \left[\frac{2\sqrt{t_1 t_2}}{(t_1+t_2)}\right]^{1/\beta}. \quad (33)$$

Making the usual transformation $\ln t = T$, Eq. (33) becomes

$$\bar{C}(T_1,T_2) = \left[\frac{2}{e^{1/2(T_1-T_2)}+e^{-1/2(T_1-T_2)}}\right]^{1/\beta} = \left[\operatorname{sech}\left(\frac{T_1-T_2}{2}\right)\right]^{1/\beta} \equiv f(|T_1-T_2|). \quad (34)$$

The correlator in Eq. (34) is now stationary since it depends only on the difference $|T_1-T_2|$ and non-Markovian.

To determine the persistence exponent we adapt the method of IIA as explained in Ref. [2]. The basic assumption is that the intervals between the successive zeros of $X(T)$ are statistically independent. We will briefly outline the method here. The first step is to construct the variable $\sigma = \operatorname{sgn}(X)$. The correlator $A(T)=\langle \sigma(T)\sigma(0)\rangle$ is given by

$$A(T) = \frac{2}{\pi}\arcsin[f(T)]. \quad (35)$$

If $p_n(T)$ is the probability that an interval of size T contains n zeros of $X(T)$, $P(T)$ is the distribution of intervals, and $Q(T)$ is the probability that the left and right of the interval contains no zeros, then

$$p_n(T) = \langle T\rangle^{-1}\int_0^T dT_1 dT_2 \dots dT_n Q(T_1) \times P(T_2-T_1)\dots P(T_n-T_{n-1})Q(T-T_n) \quad (36)$$

together with

$$A(T) = \sum_{n=0}^{\infty}(-1)^n p_n(T), \quad (37)$$

where $\langle T\rangle = -2/A'(0)$. Taking a Laplace transform of Eq. (36) and using the fact that $\tilde{P}(s)=1-s\tilde{Q}(s)$, $\tilde{P}(s)$ and $\tilde{Q}(s)$ being the Laplace transform of $P(T)$ and $Q(T)$, respectively, we arrive at

$$p_n(s) = \frac{1}{\langle T\rangle s^2}[1-\tilde{P}(s)]^2\tilde{P}^{n-1}(s) \quad \text{for } n \geqslant 1$$
$$= \frac{1}{\langle T\rangle s^2}[\langle T\rangle - 1 + \tilde{P}(s)] \quad \text{for } n=0. \quad (38)$$

Finally, substituting Eq. (38) into Eq. (37) and carrying out the summation over n yields $\tilde{P}(s)$ in terms of $\tilde{A}(s)$, that is,

$$\widetilde{P}(s) = [2 - F(s)]/F(s), \tag{39}$$

where $F(s)$ is given by

$$F(s) = 1 + \frac{\langle T \rangle s}{2}[1 - s\widetilde{A}(s)], \tag{40}$$

$\widetilde{A}(s)$ being the Laplace transform of $A(T)$. For large T, $p_0(T) \sim e^{-\theta T}$ means that the exponent θ is given by the pole of $\widetilde{P}(s)$ or the zero of $F(s)$.

In our present case, the correlator $A(T)$ is given by

$$A(T) = \frac{2}{\pi} \arcsin[\text{sech}(T/2)]^{1/\beta}, \tag{41}$$

which gives $\langle T \rangle = \pi\sqrt{4\beta}$. The function $F(s)$ has the form

$$F(s) = 1 + \pi\sqrt{\beta}s\left(1 - \frac{2}{\pi}s\int_0^\infty dTe^{-sT} \arcsin\{[\text{sech}(T/2)]^{1/\beta}\}\right). \tag{42}$$

The zeros of the function $F(s)$ are determined numerically. As a check for numerical verification we took the values $\alpha = -3$, -1, and $-1/3$, which corresponds to $1/\beta = 1/2$, 1, and $3/2$, respectively. These values of $1/\beta$ correspond to the normal diffusion in $D = 1, 2$, and 3, respectively. The exponents determined numerically using these three values of $1/\beta$ agrees well with the exponents reported in Ref. [2]. Finally, we have taken various values of α between 0 and 1 and have obtained the roots of $F(s)$ numerically. The obtained values of the exponents are listed in Table I.

TABLE I. The values of the exponents.

α	$1/\beta = 2/(1-\alpha)$	θ
0.1	2.22	−0.29341041
0.2	2.50	−0.312802995
0.3	2.86	−0.336107784
0.4	3.33	−0.364881293
0.5	4.00	−0.401726555
0.6	5.00	−0.451442543
0.7	6.67	−0.524308324
0.8	10.00	−0.64860239

In conclusion, we have considered the persistence phenomenon in advected passive scalar equation. In one dimension the velocity is drawn from a random distribution with $\langle \tilde{v}(k,\omega)\tilde{v}(-k,-\omega)\rangle \sim |k|^{-(2+\alpha)}$. The effect of the nonlinearity is to replace the zeroth order Green's function by its general form $G^{-1} = -i\omega + Dk^2 + \Sigma$, with $\Sigma \sim k^\beta$. We have determined the scaling form of Σ using self-consistent theory, which gives $\beta = (1-\alpha)/2$. Thus, in $\mathcal{O}(\epsilon^2)$, the effect of the nonlinearity is to change the original dynamics with an effective equation where the diffusion term due to molecular viscosity is replaced by a term of the form $\Sigma_0 k^\beta$. We have calculated the two time correlation for the effective process which has the form $[\text{sech}(T/2)]^{1/\beta}$. Finally, we have used IIA to calculate the persistence exponents.

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[1] For a brief review in persistence see S. N. Majumdar, Curr. Sci. **77**, 370 (1999).
[2] S. N. Majumdar, C. J. Cire, A. J. Bray, and S. J. Cornell, Phys. Rev. Lett. **77**, 2867 (1996).
[3] B. Derrida, A. J. Bray, and C. Godrèche, J. Phys. A **27**, L357 (1994).
[4] B. Derrida, V. Hakim, and V. Pasquier, Phys. Rev. Lett. **75**, 751 (1995).
[5] J. Krug, H. Kallabis, S. N. Majumdar, S. J. Cornell, A. J. Bray, and C. Sire, Phys. Rev. E **56**, 2702 (1997).
[6] B. Lee and A. D. Rutenberg, Phys. Rev. Lett. **79**, 4842 (1997).
[7] H. Kallabis and J. Krug, Europhys. Lett. **45**, 20 (1999).
[8] S. N. Majumdar and C. Sire, Phys. Rev. Lett. **77**, 1420 (1996).
[9] S. N. Majumdar and A. J. Bray, Phys. Rev. Lett. **81**, 2626 (1998).
[10] A. Watson, Science **274**, 919 (1996).
[11] M. Marcos-Martin, D. Beysens, J.-P. Bouchand, C. Godrèche, and I. Yekutieli, Physica D **214**, 396 (1995).
[12] B. Yurke, A. N. Pargellis, S. N. Majumdar, and C. Sire, Phys. Rev. E **56**, R40 (1997).
[13] W. Y. Tam, R. Zeitak, K. Y. Szeto, and J. Stavans, Phys. Rev. Lett. **78**, 1588 (1997).
[14] S. N. Majumdar, A. J. Bray, S. J. Cornell, and C. Sire, Phys. Rev. Lett. **77**, 3704 (1996).
[15] Y. G. Sinai, Theor. Math. Phys. **90**, 219 (1992).
[16] T. W. Burkhardt, J. Phys. A **26**, L1157 (1993).
[17] G. Manoj and P. Ray, Phys. Rev. E **62**, 7755 (2000).
[18] B. Luki, S. Jeney, C. Tischer, A. J. Kulik, L. Forro, and E. L. Florin, Phys. Rev. Lett. **95**, 160601 (2005).
[19] S. N. Majumdar and A. J. Bray, Phys. Rev. Lett. **91**, 030602 (2003).
[20] M. Constantin, C. Dasgupta, P. P. Chatraphorn, S. N. Majumdar, and S. Das Sarma, Phys. Rev. E **69**, 061608 (2004).
[21] S. N. Majumdar and D. Das, Phys. Rev. E **71**, 036129 (2005).
[22] S. Bhattacharya, D. Das, and S. N. Majumdar, Phys. Rev. E **75**, 061122 (2007).
[23] D. Chakraborty, Eur. Phys. J. B **64**, 263 (2008).
[24] S. Jain and T. Yamano, Int. J. Mod. Phys. C **19**, 161 (2008).
[25] J. K. Bhattacharjee and S. Bhattacharjee, Non Linear Dynamics Near and Far From Equilibrium (Hindustan Publishing House, Delhi, 2007).